On perfect colorings of infinite multipath graphs

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SECTION 1

PROBLEM STATEMENT

Lexicographical graph product

Definition

A lexicographical product of graphs G and H is a graph $G \cdot H$ with the following properties:

- $V(G \cdot H) = V(G) \times V(H)$,
- $\{(u_1, u_2), (v_1, v_2)\} \in E(G \cdot H) \Leftrightarrow \{u_1, v_1\} \in E(G)$ or $(u_1 = v_1$ and $\{u_2, v_2\} \in E(H))$.

Lexicographical graph product. Example 1

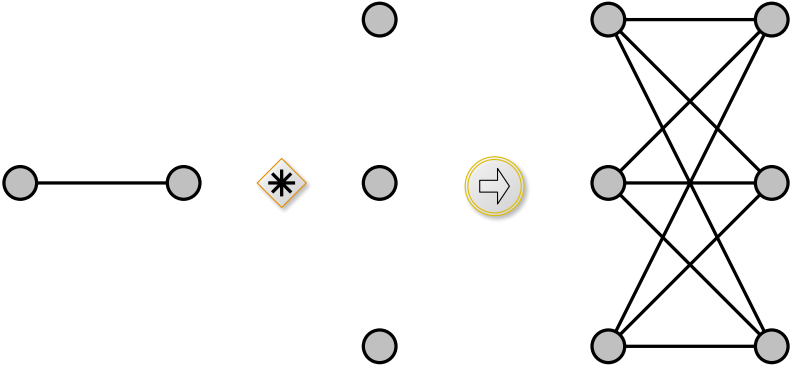


Рис.: $P_2 \cdot \overline{K_3} = K_{3,3}$

Lexicographical graph product. Example 2

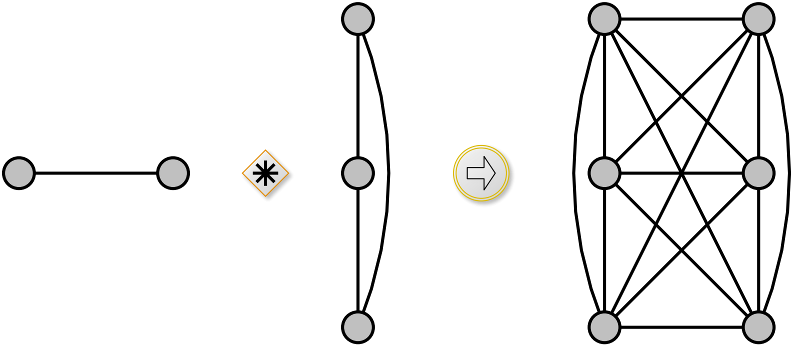


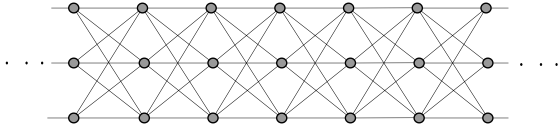
Рис.: $P_2 \cdot K_3$

Object of the research

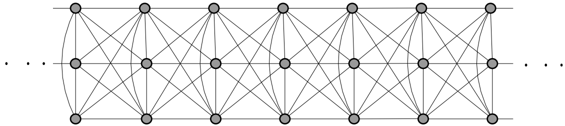
- A graph C_∞: $V(C_\infty) = \mathbb{Z}, E(C_\infty) = \{\{i, i+1\} | i \in \mathbb{Z}\}$.
- $n \in \mathbb{N}$
- A graph $\overline{K_n}$ is an empty graph on n vertices and a graph K_n is a complete graph on n vertices.
- *An infinite G-times path* is the graph $C_\infty \cdot G$.

The infinite multipath graphs. Examples

The infinite $\overline{K_3}$-times path



The infinite K_3-times path



Perfect coloring (equitable partition, partition design)

Definition

- $k \in \mathbb{N}$
- A vertex coloring of a graph $G = (V, E)$ with k colors: $\phi : V \to \{1, 2, \ldots, k\}$.
- If $\phi(v) = s$ then s is the *color* of v.
- A coloring $\phi(v)$ is called *perfect* with parameter matrix $M = (m_{ij})$ if for each vertex of color i the number of adjacent vertices of color j is equal to m_{ij}.

Perfect coloring of the graph $C_\infty \cdot \overline{K_3}$. Example

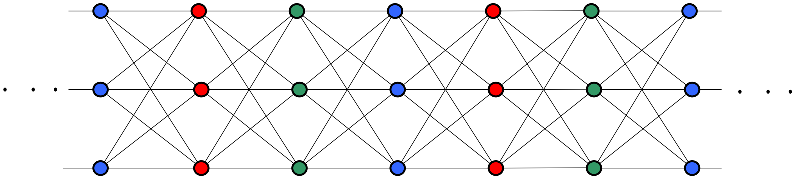


Рис.: Perfect coloring of the graph $C_\infty \cdot \overline{K_3}$

$$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix} = M$$

SECTION 2

DISJUNCTIVE COLORINGS OF THE GRAPHS $C_\infty \cdot \overline{K_n}$ AND $C_\infty \cdot K_n$

Disjunctive colorings of the graph $G \cdot H$

- A perfect coloring of the graph G:
  $\psi : V(G) \to \{1, 2, \dots k\}$;
- a set of perfect colorings of the graph H:
  $\Phi = \{\phi_1, \phi_2, \dots \phi_k\}$,
  $\phi_s : V(H) \to J_s, s \in \{1, 2, \dots k\}$,
  $J_p \cap J_q = \varnothing$ if $p \neq q$.
- $\psi \cdot \Phi(v_1, v_2) = \phi_{\psi(v_1)}(v_2)$
- This design of the graph $G \cdot H$ is called *disjunctive coloring*.

Disjunctive coloring of the graph $C_\infty \cdot \overline{K_3}$. Example

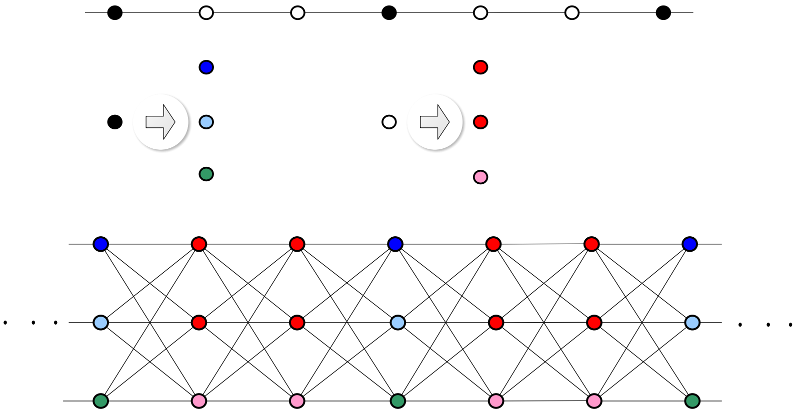


Рис.: Disjunctive coloring of the graph $C_\infty * \overline{K_3}$

SECTION 3

NONSTANDARD COLORINGS OF THE GRAPHS $C_\infty \cdot \overline{K_n}$ AND $C_\infty \cdot K_n$

Perfect colorings of a bipartite graph. Definitions

- The graph $C_\infty \cdot \overline{K_n}$ is bipartite.
- Consider a bipartite graph $G(V_1, V_2)$.
- A *half-coloring* of the graph G is a coloring of vertices of the set V_i $(i \in \{1, 2\})$.
- A perfect coloring $\phi(v)$ of the graph G is called *bipartite* if $\phi(V_1) \cap \phi(V_2) = \varnothing$.
- Otherwise the perfect coloring of the graph G is called *nonbipartite*.
  If G is connected then $\phi(V_1) = \phi(V_2)$ in nonbipartite case.

Nonstandard bipartite colorings of $C_\infty \cdot \overline{K_n}$

- A result of combination of two arbitrary 2-periodic half--colorings of the graph $C_\infty \cdot \overline{K_n}$ with disjoint sets of colors is perfect coloring of it.

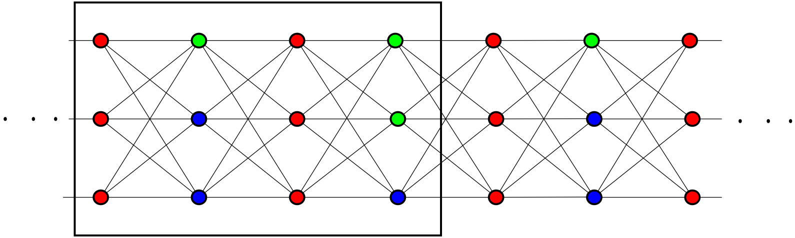
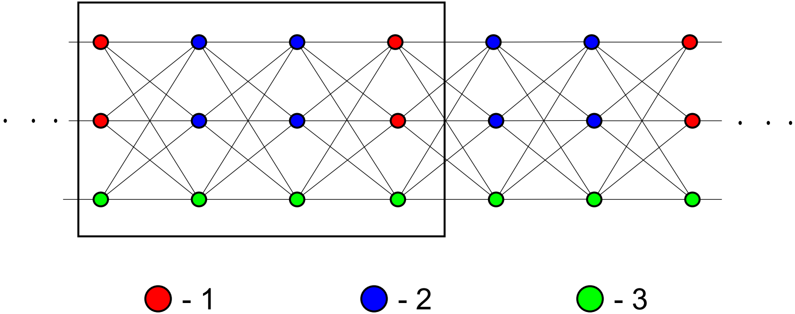


Рис.: Nonstandard bipartite coloring of the graph $C_\infty \cdot \overline{K_3}$

Nonstandard nonbipartite colorings of the graph $C_\infty \cdot \overline{K_n}$

1. We consider a coloring $\phi(v)$ of the graph $C_\infty \cdot \overline{K_n}$ with period T of length 4.
2. Let i be a color of $\phi(v)$.
3. *i-characteristic vector* b_i is a vector of length 4, whose components are numbers of i-colored vertices in corresponding copies of $\overline{K_n}$.
4. If the sum of components of the vector b_i with even numbers is equal to the sum of components with odd numbers for every i, the coloring is perfect.
5. Such the coloring of the graph $C_\infty \cdot \overline{K_n}$ is called *nonstandard nonbipartite*.

Nonstandard nonbipartite coloring of the graph $C_\infty \cdot \overline{K_3}$. Example



$b_1 = (2\ 0\ 0\ 2)^T$; $b_2 = (0\ 2\ 2\ 0)^T$; $b_3 = (1\ 1\ 1\ 1)^T$.

Nonstandard perfect colorings of the graph $C_\infty \cdot K_n$

- Any 3-periodic coloring of the graph $C_\infty \cdot K_n$ is perfect

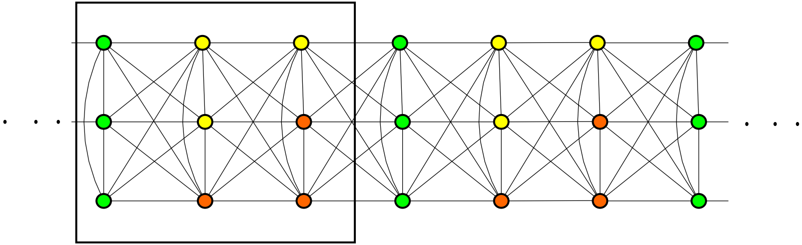


Рис.: Nonstandard perfect colorings of the graph $C_\infty \cdot K_3$

SECTION 4

RESULTS

Perfect colorings of the graph $C_\infty * \overline{K_n}$

Theorem 1

The perfect colorings of the graph $C_\infty * \overline{K_n}$ are exhausted by the following list:

1. disjunctive perfect colorings;
2. nonstandard bipartite colorings, that are got by combining two arbitrary 2-periodic half-colorings;
3. nonstandard nonbipartite colorings, that are got by combining two matched 2-periodic half-colorings.

Perfect colorings of the graph $C_\infty * K_n$

Theorem 2

The perfect colorings of the graph $C_\infty * K_n$ are exhausted by the following list:

1. disjunctive perfect colorings;
2. nonstandard 3-periodic colorings.

Thank you for your attention!